**Computational Questions**

*Least Squares Solutions*

**Find the least squares solutions of Ax=b**

Compute ATA and ATb

Solve (ATA|ATb)

**Smallest possible value of ||Ax–b||**

Compute x = least squares solution

*Projections*

**Find the projection of w onto V**

p = [w·v1/v1·v1]v1 + [w·v2/v2·v2]v2 + … + [w·vk/vk·v3k]vk

**Find the projection of w onto V**

Since w is orthogonal to V, the projection of w onto V is the zero vector.

*Planes*

**Find equation of plane that is perpendicular to V and contains w**

Plane is subspace span{w, p}

Equation of plane is ax + by + cz = 0

Q4(a)(iv) exam 1112

**Write down the equation of the plane P in the xyz-space that is not transformed to a different plane.**

Q4(a)(vi) of exam 1415sem2

**Find the equation of the plane in R3 that is transformed to the line x-y = 0 in R2 under T.**

Q5(vii) of examplify trial

*Rank Nullity*

**What is the smallest possible value of rank(B)?**

Q1(b)(iv) of exam 1415sem2

**What is the nullity of C?**

Nullity(C) = Number of Columns – rank(C)

*Dimension*

**Find all values of k such that the solution space of Bx = 0 has dimension of at least 1**

Solution space of Bx = 0 has dimension 0 if and only if Bx = 0 has only the trivial solution which is equivalent to B being invertible. Thus the values of k such that the solution space of Bx = 0 has dimension of at least 1 are k = -1 and 2.

**Let U = {u | u·v = 0 for all v}. Find a basis for and determine the dimension of U.**

Q2(a)(iii) of exam1415sem2

**Find a subspace W of R4 such that dim(W) = 3 and dim(W ∩ V) = 1**

Q2(a)(ii) of exam1415sem2

*Basis*

**Implicit form to explicit form**

Separate the parameters from the explicit form

**Find basis for span S**

rref(S), linearly independent columns

**Find basis for row space of A**

rref(A), non-zero rows of rref

**Extend basis for the row space of A**

Add pivots for zero rows

Extend basis to orthogonal basis (projection)

v – p is orthogonal to each vi, hence {v1, v2, v3, v-p} is an orthogonal basis for R4

**Extend basis for nullspace of A**

Add pivots for zero rows

**Find basis for column space of A**

rref(A), linearly independent columns of A

Q2(ii) of exam1617sem2

**Find basis for nullspace of A**

General solution of Ax = 0

Q2(iii) of exam1617sem2

**Find basis for kernel**

Ax = 0

# A is matrix of the Transformation

**Determine the basis T (given P and S)**

T’ = S’P-1

# P is the transition matrix from S to T

Q1(vi) of exam1213

*Coordinate Vector/Transition Matrix*

**Find transition matrix P, T = {v1 v2 v3} to S = {u1 u2 u3}**

Solve (u1 u2 u3 | v1 v2 v3).

# P = ([v1]S [v2]S [v3]S)

# P[u3]T = [u3]S

# [v3]S = (1 1 0)T means v3 = u1 + u2

**Find transition matrix P, S = {u1 u2 u3} to T = {v1 v2 v3}**

Solve (v1 v2 v3 | u1 u2 u3).

# P = ([u1]T [u2]T [u3]T)

# P[u3]S = [u3]T

# [u3]T = (1 1 0)T means u3 = v1 + v2

**Find transition matrix from T to S/S to T**

# S = {u, v}

# T = {u – v, u + 2v}

[u – v]S = (1 –1)

[u + 2v]S = (1 2)

T to S is P = ([u – v]S, [u + 2v]S)

S to T is P-1

**Find coordinate vector [u4]S with respect to S**

Solve (S|u4)

**Find** **coordinate vector [w]T with respect to T (given P)**

[w]T = P[w]S

**Find coordinate vector [w]S with respect to S (given P)**

[w]S = P[w]T

**Find the coordinate vector (b)S of the vector b with respect to the orthonormal basis S**

Q4(a)(v) of examplify trial

**Express v as a linear combination of the vectors in S and write down the coordinate vector**

au1 + bu2 + cu3 = v

(v)S = (a, b, c)

# [w]T = (0 1 -2), T = {v1, v2, v3}

# w = 0v1 + 1v2 – 2v3

**Find a vector w such that the coordinate vector (w)S = (a, b, c)**

w = au1 + bu2 + cu3

*Orthogonality*

**Find orthogonal matrix P such that PTAP is a diagonal matrix**

Get eigenvalues of A and obtain basis for eigenspaces

Normalize the union of the eigenspaces of A

# D = PTAP and A = PDPT

**Find two orthogonal matrices both having v1 and v2 as its first two columns**

Since u3 is orthogonal, u3 is third column

Q4(a)(v) of exam 1112

**Find orthogonal/orthonormal basis**

Gram Schmidt process

**Extend the set to an orthogonal basis for R4**

Get coefficients from implicit form

Q4(a)(iii) of exam 1213

Q4(a)(iv) of examplify trial

*Eigenvalues*

**Find characteristic polynomial/eigenvalues**

det(λI – A) = 0

**Find all eigenvalues of B and determine if B is diagonalizable**

Q3(b) of exam 1112

Q2(ii) of exam1920sem1

**Find all eigenvalues of A. Explain how you get your answer.**

Since A is an upper triangular matrix, eigenvalues are just the diagonal entries of A.

**Find basis for eigenspaces**

Solve ((λI – A)|0), find general solution

*Diagonalisation*

**Find an invertible matrix P in terms of a such that P-1AP is a diagonal matrix**

Q5(iii) of exam1617sem2

**Find an invertible matrix P and a diagonal matrix D such that B = PDP-1**

Q2(b)(i) of examplify trial

**Find an invertible matrix P and a diagonal matrix D such that A3 = PDP-1**

Q2(iv) of exam1920sem1

**Diagonalize the matrix A**

Find eigenvalues then eigenvectors and arrange them to get P

D = P-1AP

A = PDP-1

*Linear Transformations*

**Let T be a linear transformation with A as a standard matrix. Find T((1 2 3 0 0)).**

# T((1 0 0 0 0)) = (2 1 3 2)

# T((0 1 0 0 0)) = (0 5 1 0)

# T((0 0 1 0 0)) = (1 1 1 1)

T((1 2 3 0 0)) = T((1 0 0 0 0)) + 2T((0 1 0 0 0)) + 3T((0 0 1 0 0))

T((1 2 3 0 0)) = (2 1 3 2) + 2(0 5 1 0) + 3(1 1 1 1) = (5 14 8 5)

**Write down the standard matrix for T where T(x y) = (2x x+y x-y)**

(2 1 1, 0 1 -1)

**Find the standard matrix for T**

A(columns of pre-transform) = (columns of post-transform)

A = (columns of post-transform)(columns of pre-transform)-1

**Write down the standard matrix for T**

Q5(ii) of examplify trial

**Find the kernel of T**

Ax = 0

**Find the range of T**  
Ax = b

**Find a linear transformation S: R3 🡪 R3 such that (S o T)(v1) = 4v1, (S o T)(v2) = 4v2, (S o T)(v3) = –4v3**

Q4(a)(vii) of exam1415 sem2

**Let T: R3 🡪 R3 be the linear transformation with standard matrix A. Find R(T) and ker(T).**

Since A is invertible, column space of A is R3 and nullspace of A is the zero space.

R(T) = R3 and ker(T) = {0}

**Formula for the composition of S o T**

Compute ST

**Write the standard matrix of T and find ker(T) explicitly**

Q5(a) of exam1516sem2

**Find the standard matrix A for T in terms of v1, v2, v3**

Q4(ii) of exam1920sem1

**Suppose T(v1) = 2v1, T(v2) = 3v2, T(v3) = 5v3. Find v1, v2, v3.**

Q4(iv) of exam1920sem1

**Find all the vectors v in R3 such that (S o T)(v) = v, where S is the linear transformation in (v). Show your working.**

Q4(vi) of examplify trial

*Miscellaneous*

**Let M be a non-invertible 3x3 symmetric matrix such that M(1 1 0) = (2 2 0) & M(1 -1 1) = (-1 1 -1)**

Q4(c) of exam 1415sem1

**Find a 5x5 matrix without zero rows or repeating rows that has the same row space as A**

Use the non-zero rows of A, together with other rows that are linear combinations

**Find elementary matrices E1…EK such that A = EK…E1R where R is in RREF**

Q1(a)(ii) of exam 1415sem2

# first elementary matrix is the final elementary row operation

**Compute det(B)**

Use cofactor expansion

**Find all values of k such that Bx = 0 has only the trivial solution**

Bx = 0 has only the trivial solution if and only if B is invertible.

det(B) = (k-2)(k+1), B is invertible for all k, k ≠ -1, 2

**Are there values of k such that the solution space of BTx = 0 is a plane in R3 that contains the origin?**

Q1(b)(v) of exam 1415sem2

**Let n be a positive integer. Write Bn in the form (b11 b12, b21 b22) where the entries bij are in terms of n.**

Q2(b)(ii) of examplify trial

**Proving Questions**

*Row Space/Column Space/Nullspace*

**Find a non-zero vector that is contained in both the row space and the nullspace of A.**

Q1(v) of examplify trial

**Is S a basis for the column space of A?**

Since S is linearly independent and rank(A) = 3, S is a basis for column space.

**Prove that the nullspace of B is a subspace of the nullspace of AB.**

Q6(b)(i) of exam1617sem2

**By pre-multiplying A with an invertible 4x4 matrix B, is it necessary that BA has the same row space as A?**

B is invertible hence is a product of elementary matrices EN…E1

So A is row equivalent to BA, and row space is equal.

**Given nullspace, first two columns of A are linearly independent and the second and fourth columns of A are identical.**

Q6(a)(i)(ii) of exam 1516sem2

**If A is a square matrix, then its row space is equal to its column space.**

Q6(a) of exam 1415sem1

**Prove nullity(A) + nullity(B) ≥ nullity(AB)**

Q6(b)(ii) of exam1617sem2

*Invertible/Singular*

**Is A invertible? Justify your answer.**

Since det(A) ≠ 0, it is invertible.

**Is it possible to find a matrix B such that AB is an invertible matrix?**

Q1(vi) of examplify trial

**Let A be a square matrix of order n. Let Mij be the matrix of order n - 1 obtained from A by deleting the ith row and the jth column.**

Prove that if A is invertible, then at least n of the matrices Mij are invertible. [Hint: Consider

the adjoint matrix adj(A).]

Q6(a) of exam1617sem2

**Show that every invertible matrix can be written as A = BC where B is an orthogonal matrix and C is an upper triangular matrix**

Q4(c) of exam 1112

*Linear Independence*

**Show that S is linearly independent**

Consider matrix form using vectors in S as rows

Since all rows are non-zero rows, S is linearly independent.

**Given vector space, no two vectors are linearly dependent and dim V ≥ 3.**

Q6(b)(i)(ii) of exam 1516sem2

**Suppose v1, v2 and v3 are linearly independent. Show that ker(T) = {0}**

Q4(iii) of exam 1920sem1

*Basis*

**Show S is a basis for R3**

det(S) ≠ 0, S is basis for R3.

**Show that S is an orthogonal set/basis**

Dot product one by one (set & basis)

Dimension is equal to column space of V(basis)

**Let {u1, u2, u3, u4} be a basis for R4. Suppose U1 = span{u1 u2} and U2 = span{u3 u4}. Is it possible that U1 ∪ U2 = R4?**

Q2(c) of exam 1415sem1

**Suppose the set {v1 v2 v3} is an orthonormal basis for R3. Show that the set is also an orthonormal basis.**

Q3(c) of exam 1415sem1

*Coordinate Vectors*

**Is there any non-zero vector v such that (v)T = (v)S?**

Q3(a)(iv) of examplify trial

**Prove that [ku4]s = k[u4]s (coordinate vector)**

Q1(a)(iii) of exam 1112

*Diagonalization*

**Is A diagonalizable? Justify your answer.**

For each eigenvalue the dimension of the eigenspace is equal to the multiplicity of the eigenvalue.

# dimension is the number of vectors in the eigenspace for that eigenvalue

# multiplicity is the number of the same eigenvalue

**Prove A is diagonalizable**

Q4(c)(i) of exam 1213

Q5(ii) of exam1617sem2

**Let T: Rn 🡪 Rn be a linear operator such that its standard matrix is diagonalizable.**

R(T) = R(T o T) and Ker(T) = Ker(T o T)

Q5(b) of exam1516sem2

**Is AAT orthogonally diagonalizable? Why?**

Yes. AAT is a symmetric matrix, hence is orthogonally diagonalizable.

*Eigenvalues/Eigenvectors*

**Determine which five vectors v1 to v5 are eigenvectors of A.**

Multiply A by each vector, if they have eigenvalues, they are eigenvectors

**If λ ≠ 0, show that Bx is an eigenvector of BA with eigenvalue λ**

λ ≠ 0, ABx ≠ 0, Bx ≠ 0

ABx = λx

BABx = Bλx

BA(Bx) = λBx

**If λ = 0, is Bx is an eigenvector of BA with eigenvalue λ?**

Not necessary.

Let B = 0 and x any non-zero vector. Then ABx = 0

So x be an eigenvector of AB associated with eigenvalue 0.

But Bx = 0, which is not an eigenvector

**Suppose 2 and -2 are not eigenvalues, show that A + B is singular**

Q3(c) of exam1112

**There is no 3x3 matrix of rank 2 with only 1 eigenvalue.**

Q6(d) of exam 1415sem1

**Let M and N be two n x n matrices. Suppose {v1, v2, …, vn} is a set of linearly independent eigenvectors for both M and N. Then MN = NM.**

Q4(b) of exam 1415sem2

Show that (AB)2 = 5AB

Q4(ii) of exam1516sem2

*Rank*

**Prove that rank(A) = tr(A)**

Q4(c)(ii) of exam 1213

**What is the rank of BA?**

Q4(iii) of exam 1516sem2

**Is it possible to find a full rank 5x3 matrix B such that AB = 0?**

Q3(a)(iii) of exam1415sem2

**If A is an n x n matrix such that A2 = I, then rank(I + A) + rank(I – A) = n**

Q6(a) of exam1920sem1

*Linear Transformation*

**For the linear transformation T in part(vi), is there enough information to determine its formula?**

Q2(vii) of exam 1213

**Let T: Rn 🡪 Rn be a linear transformation such that ker(T) = ker(T o T). Prove ker(T o T) = ker(T o T o T).**

Q4(b) of exam1617sem2

**Show that the linear transformation F2 is the zero transformation**

Q5(b) of exam 1415sem1

**Show that the standard matrix of T is of full rank**

Q5(c) of exam 1415sem1

**Is it true that every vector in R2 is an image under T?**

Q5(iv) of examplify trial

*Orthogonal*

**Show subspace is orthogonal to u3**

Check dot product one by one

**Prove that Au · Av = u · v, ||Au|| = ||u|| and A is orthogonal**

Q4(b) of exam 1213

**There are no orthogonal matrices A and B (of the same order) such that A2 – B = AB**

Q6(b) of exam1920sem1

*Miscellaneous*

**Is it possible to find a non-zero column vector v such that Bv = v?**

Q2(b)(iii) of examplify trial

**Ax = b is inconsistent**

Q6(e) of exam 1415sem1

**What is the dimension of the vector space V = span(S)?**

Since S is linearly independent, it is a basis for V = span(S). So dim V = 3

**Find a symmetric matrix C such that C2 = A**

M2 = D

A = PDPT = PM2PT = (PMPT)(PMPT) = C2

CT = (PMPT)T = ((PT)T MT PT) = PMPT = C

**Is it possible to find a subspace U of R4 such that V ⊆ U ⊆ R4 but V ≠ U and U ≠ R4?**

Q1(vii) of exam 1213

**Find a square matrix B such that B3 = A**

A = PDP-1

B3 = A

B = PD1/3P-1

# P is a matrix of the eigenvectors

# D = P-1AP

Q3(a)(iv) of exam 1213

**Does w belong to V?**

Since projection of w onto V is equal to w itself, this implies w belongs to V.

**Suppose B is a 2x2 matrix. Find matrix C such that C2 = B.**

Q4(b) of exam 1415sem1

**Without performing Gaussian elimination, can you tell whether the system has no solution, exactly one solution or infinitely many solutions?**

Q3(b)(iii) of exam 1415sem2

**Find BA. Show how you derive your answer.**

Q4(iv) of exam 1516sem2

**Consider the linear system and determine the conditions on the constant a such that the linear system has exactly one solution, no solution, infinitely many solutions.**

Q1(a)(i)(ii)(iii) of exam1617sem2

**Is it possible to find a one-dimensional subspace of V that does not contain any column of A?**

Q1(iv) of exam1920sem1

**Show Ax = b is inconsistent**

Q3(i) of exam 1920sem1

Q4(a)(i) of examplify trial

**Suppose the first column of A is (1 0 -2). Do we have enough information to determine the matrix A? Why?**

Q5(v) of exam1920sem1

**For an m x n matrix A and m x 1 matrix b, let p be the projection of b onto the column space of A. Show that b-p is a solution of ATx = 0.**

Q4(b) of examplify trial

**The set W is not a subspace of R4**

Q6(b) of exam 1415sem1

**If span{u} ∩ span{v} = span{u, v}, then span{u} = span{v}**

Q6(c) of exam 1415sem1

**Given any 2x3 matrix M and 2x1 column vector c, the linear system Mx = c always has infinitely many least squares solutions.**

Q3(c)(ii) of exam1415sem2

**Let U and V be two subspaces of R4 such that dim U = 2 and dim V = 3. Determine whether each of the following statements is true or false.**

dim(U ∩ V) ≥ 1

**If U is not a subset of V, then dim(U ∩ V) = 1**

Q3(b)(i)(ii) of examplify trial

**MATLAB**

Size of A: size(A)

(*i*, *j*)-entry of A: A(i, j)

Zero matrix of size *m* x *n*: zeros(m, n)

Identity matrix of order *n*: eye(n)

Diagonal matrix with diagonal entries: diag([a1…an])

Extract *i*th row: A(i,:)

Extract *i*th and *j*th row: A([i,j],:)

Multiplying the *i*th row by a non-zero constant *c*: A(i,:) = c\*A(i,:)

Interchanging the *i*th and *j*th rows: A([i,j],:) = A([j,i],:])

Adding *c* times of the *j*th row to the *i*th row: A(i,:) = A(i,:) + c\*A(j,:)

Transpose AT: A’

Reduced row-echelon form: rref(A) or rref([A b])

Powers A*n*: A ^ n

Inverse A-1: inv(A)

Declare parameters: syms s t

Rank of A: rank(A)

Nullspace of A: null(A, ‘r’) (will give the vectors in the general solution, in column vectors)

Dot product of *u* and *v*: dot(u, v)

Norm of a vector: u = sym([1 2 3 4 5]) and norm(u)

S = {(1, 1, 1, 1), (1, 1, 0, 0), (0, 1, 1, 0)}

Orthonormal basis of A: A = sym([1 1 0; 1 1 1; 1 0 1; 1 0 0]) and orth(A)

Orthogonal basis of A: orth(A, ‘skipnormalization’)

Characteristic polynomial of A:

syms lambda (declares lambda)

charpoly(A, lambda) (creates the characteristic polynomial)

syms lambda

det(lambda\*eye(n) - A)

solve(ans) (solves the characteristic polynomial for all values of lambda)

Eigenvalues of A: eig(A)

Eigenvectors of A: c\*eye(n) - A and null(ans, ‘r’)

Diagonalizing A: inv(P) \* A \* P

A1 = sym(A)

[P1, D1] = eig(A1)